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PROJECTIVE SYSTEMS AND VALUATION THEORY

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INTRODUCTION

In this note we first characterize the projective systems F on $\mathbb{Z}^+$ for which $\varprojlim^{(1)} F = 0$. This completes a result of the author ((1)). Then we turn to the problem of finding topological conditions on F in order that $\varprojlim^{(1)} F = 0$. It turns out that the conditions of the Grothendieck Mittag-Leffler theorem, Bourbaki: Topologie Générale chap. 2, are sufficient, but not necessary.

The second part of the note is concerned with the study of approximation theorems in valuation theory. We prove a Proposition, (4.3), which could be considered as an extension theorem for the Ribenboim approximation theorem.

Finally we deduce the Weierstrass product theorem showing that this theorem is, in fact, an extension of the classical approximation theorem in valuation theory.

The notations of ((2)) are used without reference.

Let Γ be an ordered set and let F be a projective system of abelian groups on Γ. Put $F = \{F_\gamma, \eta_\gamma^{\gamma'}\}_{\gamma\in\Gamma}$ and suppose Γ is directed, i.e. $\forall\, \gamma, \gamma' \in \Gamma\ \exists\, \gamma'' \in \Gamma,\ \gamma'' > \gamma, \gamma'$. Then for each $\gamma \in \Gamma$ the subgroups of F_γ, $\operatorname{im} \eta_\gamma^{\gamma'}$ for $\gamma' > \gamma$, may be considered as a base for a uniform structure on F_γ and we may consider the Hausdorff completion $\hat{F}_\gamma$ of F_γ and the canonical "imbedding" ρ_γ: $F_\gamma \longrightarrow \hat{F}_\gamma$.

It turns out that $\hat{F}_\gamma \simeq \varprojlim_{\gamma'>\gamma} F_\gamma/\operatorname{im}\eta_\gamma^{\gamma'}$ and that $\rho_\gamma : F_\gamma \to \hat{F}_\gamma$ is the homomorphism induced by the homomorphisms $\mu_\gamma^{\gamma'}: F_\gamma \longrightarrow F_\gamma/\operatorname{im}\eta_\gamma^{\gamma'}$. It can easily be shown that the abelian groups $\hat{F}_\gamma$ together with obvious homomorphisms forms a projective system $\hat{F}$ on Γ, and further, that the ρ_γ, $\gamma\in\Gamma$ induces a homomorphism of projective systems on Γ

$$\rho : F \to \hat{F}.$$

§1.

The following theorem completes our Theorem 1 ((1)) , and characterizes the projective systems F on $\mathbb{Z}^+$ for which $\varprojlim_{\mathbb{Z}^+}{}^{(1)} F = 0$.

T h e o r e m (1.1) (M i t t a g - L e f f l e r) . Let Γ be an ordered set containing a denumberable cofinal subset, and let F be a projective system of abelian groups on Γ . Then the following conditions are equivalent

(i) $\varprojlim_{\mathbb{Z}^+}{}^{(1)} F = 0$

(ii) The canonical homomorphism $\rho : F \to \hat{F}$ is onto.

P r o o f . Suppose $\varprojlim_{\mathbb{Z}^+}{}^{(1)} F = 0$ then we know from ((1)) that either F satisfies the Mittag-Leffler condition or $F_\gamma/\mathrm{im}\,\eta_\gamma \simeq \widehat{F_\gamma/\mathrm{im}\,\eta_\gamma}$ for all $\gamma \in \Gamma$. In any case we have that $\rho : F \to \hat{F}$ is onto. Suppose $\rho : F \to \hat{F}$ is onto. As $\ker \rho_\gamma = \bigcap_{\gamma' > \gamma} \mathrm{im}\,\eta_\gamma^{\gamma'}$ the projective system $\ker \rho$ is surjective, thus $\varprojlim{}^{(i)} \ker \rho = 0$ for $i \geqslant 1$. Put $K = \ker \rho$ and consider the exact sequence of projective systems on Γ

$$0 \to K \to F \to \hat{F} \to 0$$

From this, using the fact that $\varprojlim_{\Gamma}{}^{(1)} K = 0$, we find

$$\varprojlim_{\Gamma}{}^{(1)} F \simeq \varprojlim_{\Gamma}{}^{(1)} \hat{F}$$

and so we may suppose F complete Hausdorff. Now fix an element $\gamma_0 \in \Gamma$ and consider for $\gamma > \gamma_0$ the exact sequence

$$0 \longrightarrow \operatorname{im} \eta^{\gamma}_{\gamma_0} \longrightarrow F_{\gamma_0} \longrightarrow F_{\gamma_0}/\operatorname{im} \eta^{\gamma}_{\gamma_0} \longrightarrow 0$$

As $\varprojlim_{\gamma > \gamma_0} F_{\gamma_0} \simeq F_{\gamma_0} \simeq \varprojlim_{\gamma > \gamma_0} F_{\gamma_0}/\operatorname{im} \eta^{\gamma}_{\gamma_0} \simeq \hat{F}_{\gamma_0}$ we have $\varprojlim^{(i)}_{\gamma > \gamma_0} \operatorname{im} \eta^{\gamma}_{\gamma_0} = 0$ for $i \geqslant 0$.

Look at the exact sequence, for $\gamma > \gamma_0$,

$$0 \longrightarrow \ker \eta^{\gamma}_{\gamma_0} \longrightarrow F_{\gamma} \longrightarrow \operatorname{im} \eta^{\gamma}_{\gamma_0} \longrightarrow 0$$

and use $\varprojlim_{\gamma > \gamma_0}$. We obtain the exact sequence

$$0 \longrightarrow \varprojlim_{\Gamma} \ker \eta^{\gamma}_{\gamma_0} \longrightarrow \varprojlim_{\Gamma} F \longrightarrow 0 \longrightarrow \varprojlim^{(1)}_{\Gamma} \ker \eta^{\gamma}_{\gamma_0} \longrightarrow \varprojlim^{(1)}_{\Gamma} F \longrightarrow$$

$$0 \longrightarrow \quad \ldots$$

We need only prove that $\varprojlim^{(1)}_{\Gamma} \ker \eta^{\gamma}_{\gamma_0} = 0$. As Γ contains a denumberable cofinal subset we may suppose that $\Gamma = \mathbb{Z}^+$. We construct as in ((1)) , a projective system G on $\mathbb{Z}^+ \times \mathbb{Z}^+$ given by:

$$G_{(m,n)} = \ker \eta^{\max(m,n)}_{\min(m,n)}$$

Obviously G is essentially zero on $\mathbb{Z}^+ \times \mathbb{Z}^+$, in fact on the diagonal Δ of $\mathbb{Z}^+ \times \mathbb{Z}^+$ it is zero, thus $\varprojlim^{(i)}_{\mathbb{Z}^+ \times \mathbb{Z}^+} G = 0$, $i \geqslant 0$.

On the other hand we have a spectral sequence converging to $\varprojlim^{(i)}_{\mathbb{Z}^+ \times \mathbb{Z}^+} G$ given by the term

$$E_2^{p,q} = \varprojlim_{m\in \mathbb{Z}^+}{}^{(p)} \varprojlim_{n\in \mathbb{Z}^+}{}^{(q)} G_{(m,n)}$$

This gives us an exact sequence

$$0 \longrightarrow \varprojlim_{m\in \mathbb{Z}^+}{}^{(1)} \varprojlim_{n\in \mathbb{Z}^+} G_{(m,n)} \longrightarrow \varprojlim_{\mathbb{Z}^+ \times \mathbb{Z}^+}{}^{(1)} G \longrightarrow \varprojlim_{m\in \mathbb{Z}^+} \varprojlim_{n\in \mathbb{Z}^+}{}^{(1)} G_{(m,n)} \longrightarrow 0$$

As we have $\varprojlim_{\Gamma}{}^{(1)} \ker \eta_{\gamma_0}^{\gamma} \simeq \varprojlim_{\Gamma}{}^{(1)} F$, the projective system on $\mathbb{Z}^+$ given by $\varprojlim_{n\in \mathbb{Z}^+}{}^{(1)} G_{(m,n)}$ is constant, thus zero. But this means that $\varprojlim_{\Gamma}{}^{(1)} F = 0$.

Q.E.D.

C o r o l l a r y (1.2) Under the hypotheses of (1.1) $\varprojlim_{\Gamma}{}^{(i)} F = 0$ for $i \geqslant 0$ if and only if $F \simeq \hat{F}$.

C o r o l l a r y (1.3) Under the hypotheses of (1.1) we have:

$$\varprojlim_{\Gamma} F \simeq \varprojlim_{\Gamma} \ker \varrho \quad .$$

C o r o l l a r y (1.4) Under the hypotheses of (1.1) , if $\varprojlim_{\Gamma}{}^{(1)} F = 0$ then either $\varprojlim_{\Gamma} F \neq 0$ or $F \simeq \hat{F}$.

§2.

Up to now we have considered only projective systems of a b e l i a n g r o u p s . We may, however, consider projective systems of topological groups and continuous transformations, and we may ask if these, in our category of projective systems of abelian groups, possess any particularities.

We shall not go into details, but we shall prove two theorems in this direction.

Theorem (2.1) Let F be a projective system of topological abelian groups and continuous homomorphisms on the ordered set Γ. Suppose Γ contains a denumberable cofinal subset and suppose:

(i) $\forall \gamma \in \Gamma$, F_γ is complete metrizable

(ii) $\forall \gamma < \gamma'$, $\operatorname{im} \eta_\gamma^{\gamma'}$ is dense in F_γ.

Then we may conclude that $\rho : F \longrightarrow \hat{F}$ is onto.

Proof. As Γ is supposed to contain a denumberable cofinal subset we may assume $\Gamma = \mathbb{Z}^+$.

We fix an $n_o \in \mathbb{Z}^+$ and we shall prove that $\rho_n : F_n \longrightarrow \hat{F}_n$ is onto. Let $\mu_m : F_n \longrightarrow F_n / \operatorname{im} \eta_n^{m+n}$ be the canonical homomorphism, and fix an element $\hat{f} \in \hat{F}_n$. Let for each $m \in \mathbb{Z}^+$, $f_m \in F_n$ be such that $\mu_m(f_m) = \hat{f}_m$. We then have:

1) $\quad f_{m+1} - f_m \in \operatorname{im} \eta_n^{m+n} \quad$ for all $\quad m \in \mathbb{Z}^+$.

Let δ_m be the metric on F_{n+m}, and put $f_o = 0$. Since $\operatorname{im} \eta_n^{n+1}$ is dense in F_n we may find an element $g_{10} \in F_{n+1}$ such that if $g_{01} = \eta_n^{n+1}(g_{10})$ then $\delta_o(f_1 - f_o + g_{01}, 0) < \frac{1}{2}o = 1$. Put $\bar{f}_1 = f_1 + g_{01}$ then we have:

$$\mu_1(\bar{f}_1) = \mu_1(f_1) = \hat{f}_1$$

$$f_2 - \bar{f}_1 \in \operatorname{im} \eta_n^{n+1}$$

Thus we may find an element $h_{10} \in F_{n+1}$ such that $h_{o1} = \eta_n^{n+1}(h_{10}) = f_2 - \bar{f}_1$. Since all η are continuous we may, using (ii), find an element $g_{20} \in F_{n+2}$ such that if $g_{11} = \eta_{n+1}^{n+2}(g_{20})$ $g_{02} = \eta_n^{n+2}(g_{20})$ then

$$\delta_o(f_2 - \bar{f}_1 + g_{02}, 0) < \tfrac{1}{2}$$

$$\delta_1(h_{10} + g_{11}, 0) < \tfrac{1}{2}o$$

Put $\bar{f}_2 = f_2 + g_{02}$, then

$$\mu_2(\bar{f}_2) = \mu_2(f_2) = \hat{f}_2$$

$$f_3 - \bar{f}_2 \in \operatorname{im}\eta_n^{n+2}$$

Continuing this process we construct elements $\bar{f}_m \in F_n$ $h_{ij} \in F_{n+i}$, $i + j = m$, $g_{rs} \in F_{n+r}$, $r + s = m + 1$ such that:

$$\delta_r(h_{r,s-1} + g_{r,s}, 0) < \tfrac{1}{2}r \quad \text{for} \quad r + s = m$$

$$\eta_{n+i-1}^{n+i}(h_{ij}) = h_{i-1,j+1}, \quad \eta_n^{n+m}(h_{m,o}) = f_{m+1} - \bar{f}_m$$

$$\eta_{n+r-1}^{n+r}(g_{r,s}) = g_{r-1,s+1}$$

$$\mu_m(\bar{f}_m) = \mu_m(f_m) = \hat{f}_m$$

$$f_{m+1} - \bar{f}_m \in \operatorname{im}\eta_n^{n+m}$$

By construction $\bar{f} = \sum_{i=0}^{\infty} (\bar{f}_{i+1} - \bar{f}_i) = \sum_{i=0}^{\infty} (h_{o,i} + g_{o,i+1})$ exists.

Further, by construction,

$$h_j = \sum_{i=0}^{\infty} (h_{ji} + g_{j,i+1})$$

exists, $h_j \in F_{n+j}$, and we have

$$\bar{f} - \eta_n^{n+j}(h_j) = \sum_{i=0}^{j-1} \bar{f}_{i+1} - \bar{f}_i = \bar{f}_j$$

so we have

$$\mu_j(\bar{f}) = \mu_j(\bar{f}_j) = \hat{f}_j \qquad \text{for all} \qquad j \in \mathbb{Z}^+ \ .$$

This means that $\rho_n(\bar{f}) = \hat{f}$ and the proof is complete.

Q.E.D.

Corollary (2.2) Under the hypotheses of (2.1) we have $\varprojlim_{\Gamma}^{(1)} F = 0$.

Corollary (2.3) Let F be a projective system of topological abelian groups and continuous homomorphisms on Γ , and suppose Γ contains a denumberable cofinal subset. Suppose further that:

(i) $\forall \gamma \in \Gamma$, F_γ is complete metrizable,

(ii) $\forall \gamma \in \Gamma \ \exists \gamma' > \gamma$ such that for all $\gamma'' > \gamma'$
im $\eta_\gamma^{\gamma''}$ is dense in im $\eta_\gamma^{\gamma'}$.

Then $\varprojlim_{\Gamma}^{(1)} F = 0$.

Proof. We may assume $\Gamma = \mathbb{Z}^+$. By the method of ((1)) we construct a function $f\colon \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ such that $f(n) \geqslant n$ for all $n \in \mathbb{Z}^+$, and such that $m \geqslant f(n)$ implies imη_n^m dense in im$\eta_n^{f(n)}$, and we prove that

$$\varprojlim_{\mathbb{Z}^+}{}^{(i)} F \simeq \varprojlim_{\mathbb{Z}^+}{}^{(i)} \overline{\operatorname{im} \eta_n^{f(n)}} \quad .$$

Put $G_n = \overline{\operatorname{im} \eta_n^{f(n)}}$, then obviously the projective system G satisfies the hypotheses in (1.1) . Consequently $\varprojlim_{\Gamma}{}^{(1)} F = \varprojlim_{\mathbb{Z}^+}{}^{(1)} G = 0$.

Q.E.D.

R e m a r k . We know (Bourbaki: Topologie Générale, chap. 2 §3 no 5) that the hypotheses of (2.3) implies $\varprojlim_{\Gamma} F \neq (0)$ if and only if F is not essentially zero. Thus, in this case we find that $\rho : F \longrightarrow \hat{F}$ is onto and $\ker \rho \neq (0)$.

§3.

Let K be a field and let Δ be the ordered set of non-archimedian valuations on K . If $v \in \Delta$ we put

A_v	valuation ring of v
m_v	maximal ideal of A_v
U_v	multiplicative groups of v-units
Γ_v	valuation group of v
K^*	multiplicative group of non-zero elements of K
$\rho_v : K^* \longrightarrow \Gamma_v$	canonical group homomorphism.

We shall study the exact sequence of projective systems of abelian groups on Δ

$$(1) \longrightarrow U \xrightarrow{i} K^* \xrightarrow{\varrho} \Gamma \longrightarrow (0)$$

If Δ_1 is a subset of Δ we put

$$U_{\Delta_1} = \varprojlim_{\Delta_1} U \,,\quad A_{\Delta_1} = \varprojlim_{\Delta_1} A$$

$$\varrho_{\Delta_1} : K^* \longrightarrow \varprojlim_{\Delta_1} \Gamma \,,\quad \phi(\Delta, \Delta_1) = \{ v / v \in \Delta \,,\ \hat{v} \cap \Delta_1 = \{0\} \}$$

Now, look at the following conditions on a subset Δ_0 of Δ, $\hat{\Delta}_0 = \Delta_0$.

1) The weak approximation theorem holds on Δ_0, i.e.

$$\forall\, \Delta_1 \subseteq \Delta_0,\ \hat{\Delta}_1 \ \text{finite},\ \forall\, \gamma \in \varprojlim_{\hat{\Delta}_1} \Gamma \,,\ \exists\ x \in K^* \,,\ \varrho_{\Delta_1}(x) = \gamma .$$

2) The weak interpolation theorem holds on Δ_0, i.e.

$$\forall\, \Delta_1 \subseteq \Delta_0,\ \hat{\Delta}_1 \ \text{finite},\ \forall\, \gamma \in \varprojlim_{\hat{\Delta}_1} \Gamma \,,\ \exists\ x \in K^* \,,\ \varrho_{\Delta_1}(x) = \gamma$$

and $v(x) = 0 \quad \forall\ v \in \phi(\Delta_0, \Delta_1)$.

3) The strong interpolation theorem holds on Δ_0, i.e.

$$\forall\, \gamma \in \varprojlim_{\Delta_0} \Gamma \quad \exists\ x \in K^* ,\quad \varrho_{\Delta_0}(x) = \gamma .$$

Obviously 3) $\Rightarrow$ 2) $\Rightarrow$ 1) and we know from ((2)) that 1) is always true and that 3) is equivalent to

3') $\varprojlim_{\Delta_0}{}^{(1)} U = 0$.

We are now going to characterize the subsets Δ_0 for which the weak interpolation theorem holds, by homological methods.

L e m m a (3.1) The following conditions on the subset Δ_o of Δ are equivalent:

(i) The weak interpolation theorem holds on Δ_o

(ii) If $\Delta_1 = \hat{\Delta}_1$ is a finite subset of Δ_o then the homomorphism $\varprojlim_{\Delta_o}{}^{(1)} U \to \varprojlim_{\phi(\Delta_o,\Delta_1)}{}^{(1)} U$ is an isomorphism.

P r o o f . Look at the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim_{\Delta_o} U \arrow[r] & K^{*} \arrow[r] & \varprojlim_{\Delta_o} \Gamma \arrow[r, "m"] & \varprojlim_{\Delta_o}{}^{(1)} U \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \varprojlim_{\Delta_o/\phi(\Delta_o,\Delta_1)} \Gamma \arrow[r, "n"] \arrow[u, "l"'] & \varprojlim_{\Delta_o/(\Delta_o,\Delta_1)}{}^{(1)} U \arrow[r] \arrow[u, "k"] & 0
\end{tikzcd}$$

in which the sequences are exact.

Now (i) is trivially equivalent to $m \circ l = 0$ for all finite subset $\Delta_1 = \hat{\Delta}_1$ of Δ_o. By commutativity this is equivalent to $k \circ n = 0$, and since n is an isomorphism this is equivalent to $k = 0$.

Consider the exact sequence

$$\cdots \longrightarrow \varprojlim_{\phi(\Delta_o,\Delta_1)} U \longrightarrow \varprojlim_{\Delta_o/\phi(\Delta_o,\Delta_1)}{}^{(1)} U \xrightarrow{\ k\ } \varprojlim_{\Delta_o}{}^{(1)} U \longrightarrow \varprojlim_{\phi(\Delta_o,\Delta_1)}{}^{(1)} U \longrightarrow 0 \cdots$$

and the lemma follows immediately.

Q.E.D.

We shall also consider the following condition on a couple of subsets $\Delta_1 \subseteq \Delta_o$, $\Delta_1 = \hat{\Delta}_1$, $\Delta_o = \hat{\Delta}_o$.

4) The interpolation theorem holds on Δ_o with respect to Δ_1, i.e.

$\forall\ \gamma \in \varprojlim_{\Delta_1} \Gamma \quad \exists \quad x \in K^*, \qquad \rho_{\Delta_1}(x) = \gamma \qquad$ and $\qquad v(x) =$

$0 \quad \forall \quad v \in \phi(\Delta_o, \Delta_1)$.

Using the proof of (3.1) we easily see that 4) is equivalent to

4') $\varprojlim_{\Delta_o}{}^{(1)} U \longrightarrow \varprojlim_{\phi(\Delta_o,\Delta_1)}{}^{(1)} U$ is an isomorphism.

E x a m p l e . If D is a unique factorisation domain and K its quotient field, then if Δ_o is the set of valuations on K finite on D we know that the weak interpolation theorem holds on Δ_o .

§4.

Suppose we are given a sequence of fields $K_o \supseteq K_1 \supseteq K_2 \supseteq \ldots \supseteq K_i \supseteq \ldots$ and denote by Δ_i the set of all discrete valuations on K_i , and by U_i the projective system of units on Δ_i . We then have an inductive system of ordered sets $\Delta_o \longrightarrow \Delta_1 \longrightarrow \Delta_2 \longrightarrow \ \ldots \ \longrightarrow \Delta_i \longrightarrow \ \ldots$. Let K be the intersection of the fields K_i , let Δ be the set of discrete valuations on K , and let U be the projective system of units on Δ . We have

$$\Delta = \varinjlim_{\mathbb{Z}^+} \Delta_i \quad .$$

For each $i \in \mathbb{Z}^+$ let η^i be the canonical map $\Delta_i \longrightarrow \Delta$, let $\eta^i_{i+1} : \Delta_i \longrightarrow \Delta_{i+1}$ be the map corresponding to the inclusion $K_i \supseteq K_{i+1}$, and let $\bar{\Delta}_i$ be a subset of Δ_i such that $\hat{\bar{\Delta}}_i = \bar{\Delta}_i$,

$\eta^{i}_{i+1}(\bar{\Delta}_i) \subseteq \bar{\Delta}_{i+1}$. Put $\bar{\Delta} = \varinjlim \bar{\Delta}_i$, then $\bar{\Delta} = \hat{\bar{\Delta}}$ is a sub-set of ...

Let $\Delta' = \hat{\Delta}'$ be a subset of $\bar{\Delta}$ and put $\Delta'_i = (\eta^i)^{-1}(\Delta')$. Associated with η^{i}_{i+1} we have a $\mathcal{H}$ -functor

$$\mathcal{H}^{i+1}_{i} : \bar{\Delta}_{i+1} \longrightarrow P\bar{\Delta}_i$$

and associated with η^i we have the $\mathcal{H}$ -functor

$$\mathcal{H}_i : \bar{\Delta} \longrightarrow P\bar{\Delta}_i$$

(see ((3))) . We also have a homomorphism of projective systems on Δ_{i+1}

1) $$U_{i+1} \longrightarrow \mathcal{H}^{i+1}_{i} \star U_i$$

Moreover we have homomorphisms

2) $$(\mathcal{H}^{i+1}_{i})^{\circ} : \varprojlim_{\mathcal{H}_{i+1}}{}^{(\cdot)} (\mathcal{H}^{i+1}_{i}) \star U_i \longrightarrow \varprojlim_{\mathcal{H}_i}{}^{(\cdot)} U_i$$

$$(\mathcal{H}^{i+1}_{i})^{\circ} : \varprojlim_{\bar{\Delta}_{i+1}}{}^{(\cdot)} (\mathcal{H}^{i+1}_{i}) \star U_i \longrightarrow \varprojlim_{\bar{\Delta}_i}{}^{(\cdot)} U_i$$

which composed with 1) gives homomorphisms

3) $$\varprojlim_{\mathcal{H}_{i+1}}{}^{(\cdot)} U_{i+1} \longrightarrow \varprojlim_{\mathcal{H}_i}{}^{(\cdot)} U_i \quad , \quad \varprojlim_{\Delta_{i+1}}{}^{(\cdot)} U_{i+1} \longrightarrow \varprojlim_{\Delta_i}{}^{(\cdot)} U_i \quad .$$

Now the $\mathcal{H}$ -functor $\mathcal{H}_i$ gives rise to a spectral sequence which in this case (Δ_i are all trees), reduces to a short exact sequence

$$(4)\qquad 0 \longrightarrow \varprojlim_{\bar{\Delta}}{}^{(1)} \varprojlim_{\mathcal{H}_i} U_i \longrightarrow \varprojlim_{\bar{\Delta}_i}{}^{(1)} U_i \longrightarrow \varprojlim_{\bar{\Delta}} \varprojlim_{\mathcal{H}_i}{}^{(1)} U_i \longrightarrow 0$$

which commutes with the homomorphisms (3) thus gives us a projective system of short exact sequences on $\mathbb{Z}^+$.

Put $\Phi = \Phi(\bar{\Delta}, \Delta')$, $\Phi_i = \Phi(\bar{\Delta}_i, \mathcal{H}^i(\Delta')) = \mathcal{H}^i(\Phi(\bar{\Delta}, \Delta'))$ and consider the commutative diagram in which the sequences are exact.

$$(5)\qquad \begin{array}{ccccccccc}
0 & \longrightarrow & \varprojlim_{\bar{\Delta}}{}^{(1)} \varprojlim_{\mathcal{H}_i} U_i & \longrightarrow & \varprojlim_{\bar{\Delta}_i}{}^{(1)} U_i & \longrightarrow & \varprojlim_{\bar{\Delta}} \varprojlim_{\mathcal{H}_i}{}^{(1)} U_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \varprojlim_{\Phi}{}^{(1)} \varprojlim_{\mathcal{H}^i} U_i & \longrightarrow & \varprojlim_{\Phi_i}{}^{(1)} U_i & \longrightarrow & \varprojlim_{\Phi} \varprojlim_{\mathcal{H}^i}{}^{(1)} U_i & \longrightarrow & 0
\end{array}$$

L e m m a (4.1) There exists a diagram of exact sequences.

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \big| & & & \\
 & & & \varprojlim_{\mathbb{Z}^+} \varprojlim_{\bar{\Delta}}{}^{(1)} \varprojlim_{\mathcal{H}i} U_i & & & \\
 & & & \uparrow & & & \\
0 \longrightarrow \varprojlim_{\bar{\Delta}}{}^{(1)} \varprojlim_{\mathbb{Z}^+} \varprojlim_{\mathcal{H}i} U_i & \longrightarrow & & G & & \longrightarrow & \varprojlim_{\bar{\Delta}} \varprojlim_{\mathbb{Z}^+}{}^{(1)} \varprojlim_{\mathcal{H}i} U_i \longrightarrow 0 \\
 & & & \uparrow & & & \\
 & & & \varprojlim_{\mathbb{Z}^+}{}^{(1)} \varprojlim_{\bar{\Delta}} \varprojlim_{\mathcal{H}i} U_i & & & \\
 & & & \uparrow & & & \\
 & & & 0 & & &
\end{array}$$

P r o o f . Straight forward calculation using the complexes $\Pi^{\cdot}$.

Q.E.D.

L e m m a (4.2) We have:

$$\varprojlim_{\mathbb{Z}^+} \varprojlim_{\mathcal{H}i} U_i = U \quad , \quad \varprojlim_{\bar{\Delta}} \varprojlim_{\mathcal{H}i} U_i = \varprojlim_{\bar{\Delta}_i} U_i$$

$$\varprojlim_{\Phi} \varprojlim_{\mathcal{H}i} U_i = \varprojlim_{\Phi_i} U_i$$

P r o o f . Very easy.

Q.E.D.

P r o p o s i t i o n (4.3) If the interpolation theorem on $\bar{\Delta}_i$ with respect to Δ'_i holds then the interpolation theorem on $\bar{\Delta}$ with respect to Δ' holds if

$$\varprojlim_{\mathbb{Z}^+}{}^{(1)} \varprojlim_{\bar{\Delta}_i} U_i \longrightarrow \varprojlim_{\mathbb{Z}^+}{}^{(1)} \varprojlim_{\Phi_i} U_i$$

is injective. If $\varprojlim_{\mathbb{Z}^+}{}^{(1)} \varprojlim_{\mathcal{H}i} U_i = 0$ then this condition is also necessary.

P r o o f . Inspection of the diagram 5) and use (4,1) and (4.2) .

Q.E.D.

R e m a r k . The product theorem of Weierstrass now follows easily. Let K be the field of meromorphic functions on an open and connected subset D of the complex plane. Let D_i , $i \in \mathbb{Z}^+$ be relatively compact open connected subsets of D such that $D = \bigcup_{i \in \mathbb{Z}^+} D_i$, $\bar{D}_i \subseteq D_{i+1}$ $i \in \mathbb{Z}^+$.

Let K_i be the field of meromorphic functions on D_i , $i \in \mathbb{Z}^+$, then we have $\supseteq K_i \supseteq K_{i+1} \supseteq \ldots \supseteq K$ and $K = \bigcap_{i \in \mathbb{Z}^+} K_i$. Let $\Delta_i = D_i$ be the set of valuations on K_i corresponding to the points in D_i , and put $\Delta = D$. Then we have an inductive system of ordered sets

$$\to \Delta_i \to \Delta_{i+1} \to \qquad \ldots \to \Delta \qquad .$$

Let Δ' be a subset of Δ such that $\Delta'_i = \Delta' \cap \Delta_i$ is finite for every $i \in \mathbb{Z}^+$. Then we know that the interpolation theorem on Δ_i with respect to Δ_i' holds (this is the obvious rational case) and so a condition for the interpolation theorem to hold on Δ with respect to Δ' is that

(6) $$\varprojlim_{\mathbb{Z}^+}{}^{(1)} \varprojlim_{\Delta_i} U_i = 0 \quad .$$

But $\varprojlim_{\Delta_i} U_i$ is the multiplicative group of units U_{Di} in the complete metrizable algebra A_{Di} of all holomorphic functions on D_i , with the topology of uniform convergence on compact subsets. Now suppose the open sets D_i are such that $A_{D_{i+1}}$ is a dense subset of A_{D_i} , $i \in \mathbb{Z}^+$. It can then be seen that U_{D_i} is all complete metrizable and that $U_{D_{i+1}}$ is a dense subset of U_{D_i} , thus (6) holds, and this implies the existence part of the Weierstrass product-theorem.

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